A Study of Quantum Entanglement through Non-local Games
Project Report for CS498A: Undergraduate Project

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Abstract

In this report, we give a brief introduction to Non-local games, quantum strategies for non-local games and some properties of special classes of non-local games like binary games and binary constraint system(BCS) games. One of the major reasons why we wish to study about non-local games is because they offer a simple model which helps us capture the advantages of allowing quantum information sharing. In fact, the celebrated Bell inequalities[1] are equivalent to upper bounds on classical strategies for non-local games.

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1 Introduction

One of the most remarkable and non-intuitive properties of Quantum Mechanics is quantum entanglement. Quantum entanglement is a uniquely quantum mechanical resource that is widely used in almost all applications of quantum computation and quantum information. Yet, there is no complete theory of entanglement and we know very little about it. A study of computational complexity theory shows us that allowing for resources like space(complexity class PSPACE) changes the complexity class immensely. It is, but a natural question to ask what change do we get if we allow for the usage of entanglement.

Non local games are relatively simple problems where we want to investigate the change in complexity if we allow for the usage of entanglement.

Note: In this report, sections 2-3 are completely based on [2]. I have only read and re-written it in a way that is more comfortable to understand for students. Interested readers are encouraged to explore [2] for more examples of non-local games and also more properties of Non-local games.

2 Non-Local Games

2.1 Structure of Non-Local games

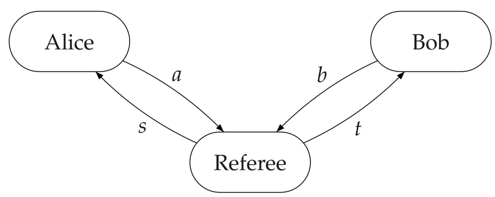

Figure 1: The communication structure of a nonlocal game[2]

Non-local games were introduced by Cleve et.al in [2]. Since then they have been extensively studied in that form(over bell inequalities) in most of the computer science literature pertaining to quantum information.

Typically, in non-local games we have 2 players Alice and Bob and a referee. The game is not a competitive game but a co-operative game. That is, Alice and Bob play together as a team.

The game starts with the referee giving one question each to Alice and Bob. Alice and Bob win the game if their answers satisfy a winning condition that is defined before the game starts. Once the game starts, Alice and Bob are not allowed to communicate between each other.

Formally a 2 player non-local game is defined by the tuple (S, T, π, A, B, V) where S is the set of all questions that are allowed to be asked to Alice. Similarly, T is of all possible questions that can be asked to Bob. π is a probability distribution over $S \times T$. Here $\pi(s_1, t_1)$ where $s_1 \in S$ and $t_1 \in T$ represents the probability for the question asked to Alice to be s_1 and the question asked to Bob to be t_1. A, B are the sets of possible answers given by Alice and Bob respectively. V is the winning condition (also referred to as the predicate) which is a function from $S \times T \times A \times B \to \{0,1\}$. If replies a, b are winning for questions s, t then $V(a, b|s, t) = 1$

2.2 Classical Deterministic Strategies

Since Alice and Bob know the probability distribution π and the winning condition V, they can decide upon a strategy before the game starts and they move apart. A *deterministic* classical strategy is basically a function from $S \to A$ and $T \to B$. Alice gets a question $s \in S$ and looks up her response for s from their predefined strategy. This is just like a function.

We define the classical value ω_c of a game as the maximum possible winning probability of a game over all possible classical strategies.

2.3 Classical Randomised Strategies

The only difference between deterministic and randomised classical strategies is that instead of the response for a question s to be a fixed value a, we have a probability distribution over A i.e a_1 with probability p_1, a_2 with probability p_2 etc such that $\sum_i p_i = 1$.

Every randomised strategy can be written as a convex combination of deterministic strategies by writing every response for s as $\sum_i p_i a_i$ and taking their product by considering the responses to be independent of each other. Thus, the classical value of a game is equal to it's winning chance for it's best deterministic strategy. In other words, we have

$$\omega_c = \max_{a,b} \sum_{s,t} \pi(s,t) V(a(s), b(t)|s,t)$$

2.4 Quantum Strategies

From here on, the reader is assumed to have a basic understanding of quantum measurements and quantum states. They can refer to section 2.2 of [4] for all the basic terminology used here.

In the classical case, Alice and Bob share a 'strategy' before they move apart from each other. What is this strategy? It is basically classical information. For a quantum strategy, we allow not only classical information to be shared between Alice and Bob but we also allow for the sharing of quantum information i.e entangled states. Also, Alice and Bob are allowed to perform quantum measurements on their parts of the entangled states after they move apart and their choice of measurements can depend on the questions which they are asked by the Referee.

A quantum strategy for a game is decided by:

- an integer $n > 0$ which captures the amount of entanglement shared between Alice and Bob.
- A unit vector $|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ which is the shared entangled state between Alice and Bob. The first $\mathbb{C}^n$ is Alice's part of $|\psi\rangle$ and the second $\mathbb{C}^n$ is Bob's part of $|\psi\rangle$.
- Sets of $n \times n$ PSD matrices $\{X_a^s : s \in S, a \in A\}$ and $\{Y_t^b : t \in T, b \in B\}$. These are the POVM elements of the measurements used by Alice and Bob respectively.

Since the PSDs are POVM elements, they satisfy

$$\sum_{a \in A} X_a^s = I_{n\times n} \quad and \quad \sum_{b \in B} Y_t^b = I_{n\times n}$$

If Alice gets the question $s \in S$ and Bob gets the question $t \in T$, then the probability with which they answer $a \in A$ and $b \in B$ is given by $\langle\psi| X_a^s \otimes Y_t^b |\psi\rangle$. Similar to the randomised strategy case(after we split the randomised strategy into a convex combination), the winning probability for a game G using a quantum strategy is

$$\sum_{s,t} \pi(s,t) \sum_{a,b} \langle\psi| X_a^s \otimes Y_t^b |\psi\rangle V(a,b|s,t)$$

Similar to the *classical* value of a game ω_c, we define the *quantum* value of a game ω_q to be the supremum of all quantum strategies. Here we consider the supremum instead of the highest because we do not know if the number n is finite.

3 Binary games

Binary games are non-local games where the answers of Alice and Bob are restricted to be bits i.e the answer sets A and B are both {0,1}. Therefore, by the definition of quantum strategies, we can see that for every $s \in S$, there are only two PSD's X_s^0 and X_s^1 which make up the measurement. Similarly for $t \in T$. Also, we have

$$X_s^0 + X_s^1 = I_{n\times n} \quad and \quad Y_t^0 + Y_t^1 = I_{n\times n}$$

for all $s \in S$ and $t \in T$.

3.1 XOR games

XOR games are a subclass of binary games where we further restrict the predicate V to be only a function of the $a \oplus b$. Since the predicate depends only on $a \oplus b$ but not on a and b individually, it is more convenient to consider the predicate as a function of only three variables $a \oplus b, s, t$. Therefore, we prefer to write V as $V(a \oplus b|s,t)$ over $V(a,b|s,t)$ in the case of XOR games.

3.2 Projective measurements

Theorem 1. *For all binary non-local games, quantum strategies with POVM measurements can be replaced with quantum strategies using only projective measurements without any loss in the winning probability.*

This significantly reduces the complexity of analysis for further properties of binary games since we can ignore the complete POVM formalism of measurements and only consider projective measurements.

Proof. Consider any quantum strategy for a binary non-local game. It consists of a shared state $|\psi\rangle$ and sets of POVM elements $\{X_s^0, X_s^1\}$ for every $s \in S$ and $\{Y_t^0, Y_t^1\}$ for every $t \in T$.
Every POVM element is a PSD and hence a normal matrix. Therefore a spectral decomposition

exists for every POVM element.
Consider the spectral decomposition of X_s^0

$$X_s^0 = \sum_{i=1}^{n} \lambda_{s,i} \left|\phi_{s,i}\right\rangle \left\langle\phi_{s,i}\right|$$

Since $X_s^0 + X_s^1 = I_{n\times n}$, we have

$$X_s^1 = \sum_{i=1}^{n} (1-\lambda_{s,i}) \left|\phi_{s,i}\right\rangle \left\langle\phi_{s,i}\right|$$

Also, we have $0 \leq \lambda_{s,i} \leq 1$ for all s, i for both of these to be valid POVM's.

Consider the POVM's where $|\phi_{s,i}\rangle$'s remain the same as in the above case but $\lambda_{s,i}$'s change. They are still valid POVM's for our binary non-local game because they satisfy the constraint of completeness. As defined above, the winning chance for our strategy for the game is

$$\sum_{s,t} \pi(s,t) \sum_{a,b} \langle\psi| X_a^s \otimes Y_t^b |\psi\rangle V(a,b|s,t)$$

Keeping Y_t^b the same, if we expand the X_0^s and X_1^s using the spectral decomposition, we can see that the above summation is a linear function of the variables $\lambda_{s,i}$.
Since $0 \leq \lambda_{s,i} \leq 1$ and this is a linear function of $\lambda_{s,i}$'s, we can see that the maximum value of this winning probability is achieved when all $\lambda_{s,i}$'s are either 0 or 1.
All POVM's for which $\lambda_{s,i}$'s is either 0 or 1 are projective measurements. Thus, we found a projective measurement as a replacement for our X_s^0 and X_s^1 for all $s \in S$. We can use the exact same line of reasoning with Y_t^0 and Y_t^1 and show that they can all be replaced by projective measurements as well. $\square$

3.3 Perfect Strategies

Theorem 2. *For a binary non-local game, if a perfect quantum strategy exists, then a perfect classical strategy exists as well.*

$$\omega_q = 1 \iff \omega_c = 1$$

4 Binary Constraint System games

This section is based on [3].
A binary constraint system is built from

- n binary variables
- m equality constraints where each constraint is on a boolean function of a subset of the n variables.

Example: We have 9 binary variables v_i, $1 \le i \le 9$ and 6 constraints

$$v_1 \oplus v_2 \oplus v_3 = 0 \qquad v_1 \oplus v_4 \oplus v_7 = 1$$
$$v_4 \oplus v_5 \oplus v_6 = 0 \qquad v_2 \oplus v_5 \oplus v_8 = 1$$
$$v_7 \oplus v_8 \oplus v_9 = 0 \qquad v_3 \oplus v_6 \oplus v_9 = 1$$

Corresponding to every binary constraint system, we can define a non-local game.
The set of questions to Alice $A = \{1, 2 \ldots \text{m}\}$
The set of questions to Bob $B = \{1, 2 \ldots \text{n}\}$
The set of answers for Alice $S = \{0,1\}^n$ where Alice gives an n-bit string assigning 0 or 1 to each of the n variables.
The set of answers for Bob $T = \{0, 1\}$ where Bob gives an assignment to his questioned variable v_i
The winning condition is that both the following conditions should hold:

- Alice's assignment should satisfy the constraint c_j where j is Alice's question
- Bob's answer should be consistent with Alice's answer i.e the $i'th$ bit in the string given by Alice should be same as Bob's answer where i is Bob's question.

For simplicity the probability distribution π is taken to be uniform.

It is simple to show that a perfect winning classical strategy can exist for a BCS game if and only if the underlying constraint system is satisfiable.

4.1 Magic square game

Consider a matrix M such that

$$M = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{bmatrix}$$

where the variables v_i are the same as those defined in the BCS example given above.
Such a matrix is a called a Magic square.
The sum of it's row entries is even for every row and the sum of it's column entries is odd for every column. On little introspection, it becomes clear that such a matrix cannot exist since XORing the left three equations and the right three equations gives us 0=1.

The non-local game associated with the above defined constraint system is called as the Magic square game and a perfect quantum strategy exists for the game i.e $\omega_q = 1$.

5 Future Work

Recently, very interesting work was done by Slofstra in [6] in which he solves one version of the Tsirelson's problem [5]. I intend to study this construction and understand the characteristics of the problem which create the difference between commuting operator and tensor product measurements in this case.

6 Acknowledgement

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